

# A NORMAL GENERATING SET FOR THE TORELLI GROUP OF A NON-ORIENTABLE CLOSED SURFACE

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**ABSTRACT.** For a closed surface  $S$ , its Torelli group  $\mathcal{I}(S)$  is the subgroup of the mapping class group of  $S$  consisting of elements acting trivially on  $H_1(S; \mathbb{Z})$ . When  $S$  is orientable, a generating set for  $\mathcal{I}(S)$  is known (see [13]). In this paper, we give a normal generating set of  $\mathcal{I}(N_g)$  for  $g \geq 4$ , where  $N_g$  is a genus- $g$  non-orientable closed surface.

## 1. INTRODUCTION

For a closed connected *non-orientable* surface  $S$ , the *mapping class group*  $\mathcal{M}(S)$  of  $S$  is defined to be the group of isotopy classes of all diffeomorphisms over  $S$ . For a closed connected *orientable* surface  $S$ , the *mapping class group*  $\mathcal{M}(S)$  of  $S$  is defined to be the group of isotopy classes of all *orientation-preserving* diffeomorphisms over  $S$ . In this paper, for  $x, y \in \mathcal{M}(S)$  the composition  $yx$  means that we first apply  $x$  and then  $y$ . The *Torelli group*  $\mathcal{I}(S)$  of  $S$  is the subgroup of  $\mathcal{M}(S)$  consisting of elements acting trivially on  $H_1(S; \mathbb{Z})$ . Let  $\Sigma_g$  be a genus- $g$  orientable closed surface. Powell [13] showed that  $\mathcal{I}(\Sigma_g)$  is generated by *BSCC maps* and *BP maps*. In [14], Putman proved Powell's result more conceptually. In addition, Johnson [7] showed that  $\mathcal{I}(\Sigma_g)$  is generated by a finite number of BP maps. In this paper, we consider the case where  $S$  is a non-orientable closed surface.

Let  $N_g$  denote a genus- $g$  non-orientable closed surface, that is,  $N_g$  is a connected sum of  $g$  real projective planes. As another classification, we see that  $N_g$  is a connected sum of a genus- $h$  orientable closed surface with  $(g - 2h)$  real projective planes, for  $0 \leq h < \frac{g}{2}$ . In this paper, we regard  $N_g$  as a surface which is obtained by attaching  $g - 2h$  Möbius bands to a genus- $h$  compact orientable surface with  $g - 2h$  boundaries for  $0 \leq h < \frac{g}{2}$  (see Figure 1). For  $R = \mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$ , let  $\cdot : H_1(N_g; R) \times H_1(N_g; R) \rightarrow \mathbb{Z}/2\mathbb{Z}$  be the mod 2 intersection form, and let  $\text{Aut}(H_1(N_g; R), \cdot)$  be the group of automorphisms over  $H_1(N_g; R)$  preserving the mod 2 intersection form. McCarthy-Pinkall [11] and Gadgil-Pancholi [5] proved that the natural homomorphism  $\rho : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}), \cdot)$  is surjective.

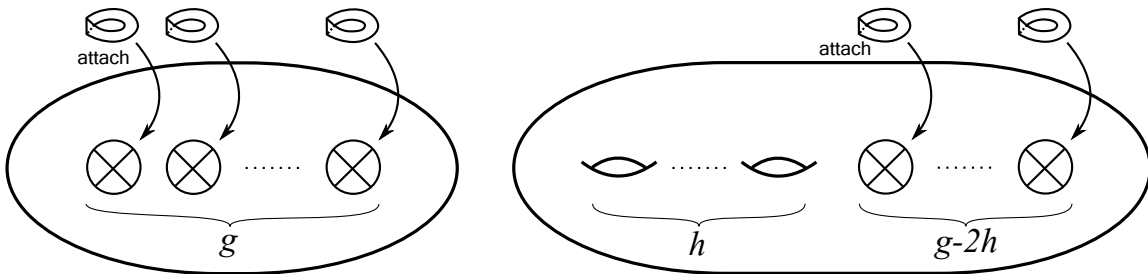


FIGURE 1. A genus- $g$  non-orientable closed surface  $N_g$ .

Lickorish [9] showed that  $\mathcal{M}(N_g)$  is generated by Dehn twists and  $Y$ -homeomorphisms. In addition, Lickorish [10] showed that the subgroup of  $\mathcal{M}(N_g)$  generated by Dehn twists is an index 2 subgroup of  $\mathcal{M}(N_g)$ . Hence  $\mathcal{M}(N_g)$  is not generated by Dehn twists. On the other hand, since a  $Y$ -homeomorphism acts on  $H_1(N_g; \mathbb{Z}/2\mathbb{Z})$  trivially,  $\mathcal{M}(N_g)$  is not generated by  $Y$ -homeomorphisms. Chillingworth [2] found a finite generating set for  $\mathcal{M}(N_g)$ . Presentations for  $\mathcal{M}(N_1)$  and  $\mathcal{M}(N_2)$  are known classically. A finite presentation for  $\mathcal{M}(N_3)$  is obtained by Birman and Chillingworth in [1]. A finite presentation for  $\mathcal{M}(N_4)$  is obtained by Szepietowski in [16]. Finally, a finite presentation for  $\mathcal{M}(N_g)$  is obtained by Paris, Szepietowski [12] and Stukow [15] for  $g \geq 4$ .

For a simple closed curve  $c$  on  $N_g$ ,  $c$  is called an  $A$ -circle (resp. an  $M$ -circle) if its regular neighborhood is an annulus (resp. a Möbius band) (see Figure 2). Let  $a$  and  $m$  be an  $A$ -circle and an  $M$ -circle on  $N_g$  respectively. Suppose that  $a$  and  $m$  intersect transversely at only one point. We define a  $Y$ -homeomorphism  $Y_{m,a}$  as follows. Let  $K$  be a regular neighborhood of  $a \cup m$  in  $N_g$ , and let  $M$  be a regular neighborhood of  $m$  in the interior of  $K$ . Note that  $K$  is homeomorphic to the Klein bottle with a boundary.  $Y_{m,a}$  is a homeomorphism over  $N_g$  which is described as the result of pushing  $M$  once along  $a$  keeping the boundary of  $K$  fixed (see Figure 3). For an  $A$ -circle  $c$  on  $N_g$ , we denote by  $t_c$  a Dehn twist about  $c$ , and the direction of the twist is indicated by a small arrow written beside  $c$  as shown in Figure 4.

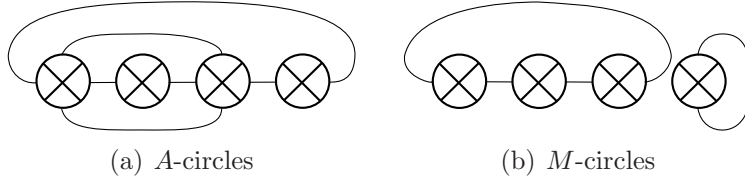
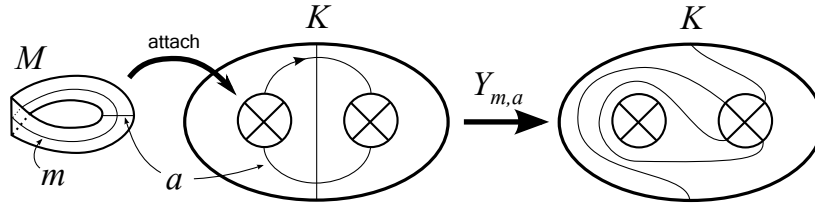
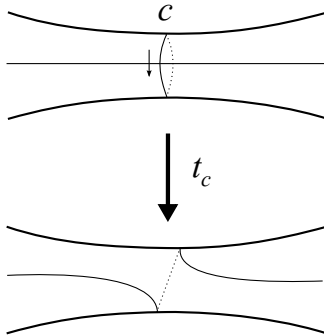


FIGURE 2.

FIGURE 3. A  $Y$ -homeomorphism  $Y_{m,a}$ .FIGURE 4. A Dehn twist  $t_c$  about  $c$ .

Let  $c$  be an  $A$ -circle on  $N_g$  such that  $N_g \setminus c$  is not connected. We call  $t_c$  a *bounding simple closed curve map*, for short a *BSCC map* (see Figure 5 (a)). Let  $c_1$  and  $c_2$  be  $A$ -circles on  $N_g$  such that  $N_g \setminus c_i$  is connected,  $N_g \setminus (c_1 \cup c_2)$  is not connected and one of its connected components is orientable. We call  $t_{c_1} t_{c_2}^{-1}$  a *bounding pair map*, for short a *BP map* (see Figure 5 (b)). In Section 2, we will see that BSCC maps and BP maps are in  $\mathcal{I}(N_g)$ .

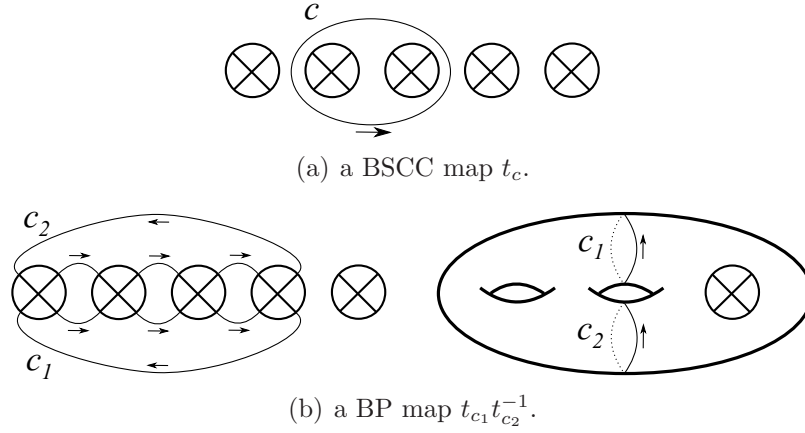


FIGURE 5.

For  $h, b \geq 1$ , let  $N_h^b$  be a non-orientable surface of genus  $h$  with  $b$  boundary components, and let  $\Sigma_h^b$  an orientable surface of genus  $h$  with  $b$  boundary components. Our main result is the following.

**Theorem 1.1.** *For  $g \geq 5$ ,  $\mathcal{I}(N_g)$  is generated by following elements.*

- BSCC maps  $t_c$  such that one of connected components of  $N_g \setminus c$  is homeomorphic to  $N_2^1$ , the other component of  $N_g \setminus c$  is non-orientable.
- BP maps  $t_{c_1} t_{c_2}^{-1}$  such that one of connected components of  $N_g \setminus (c_1 \cup c_2)$  is homeomorphic to  $\Sigma_1^2$ , the other component of  $N_g \setminus (c_1 \cup c_2)$  is non-orientable.

$\mathcal{I}(N_4)$  is generated by following elements.

- BSCC maps  $t_c$  such that one of connected components of  $N_4 \setminus c$  is homeomorphic to  $N_2^1$ , the other component of  $N_4 \setminus c$  is non-orientable.
- BSCC maps  $t_c$  such that one of connected components of  $N_4 \setminus c$  is homeomorphic to  $N_2^1$ , the other component of  $N_4 \setminus c$  is orientable.
- BP maps  $t_{c_1} t_{c_2}^{-1}$  such that one of connected components of  $N_4 \setminus (c_1 \cup c_2)$  is homeomorphic to  $\Sigma_1^2$ , the other component of  $N_4 \setminus (c_1 \cup c_2)$  is an annulus as shown in Figure 6.

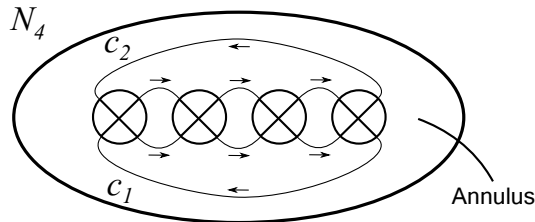


FIGURE 6.

In this theorem, these generating sets are infinite. We do not know that whether or not  $\mathcal{I}(N_g)$  can be finitely generated, and generated by only BSCC maps or BP maps.

Here is an outline of a proof of Theorem 1.1. Let  $\Gamma_2(N_g)$  be the subgroup of  $\mathcal{M}(N_g)$  consisting of elements acting trivially on  $H_1(N_g; \mathbb{Z}/2\mathbb{Z})$ . We call  $\Gamma_2(N_g)$  the *level 2 mapping class group* of  $N_g$ . Note that  $\mathcal{I}(N_g) \subset \Gamma_2(N_g)$ . Let  $\Phi_g : \text{Aut}(H_1(N_g; \mathbb{Z}), \cdot) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}/2\mathbb{Z}), \cdot)$  be the natural epimorphism. Consider the natural homomorphism  $\rho' : \Gamma_2(N_g) \rightarrow \ker \Phi_g$ . Then we have that  $\mathcal{I}(N_g)$  is equal to  $\ker \rho'$ . Let  $\Gamma_2(n) = \ker(GL(n; \mathbb{Z}) \rightarrow GL(n; \mathbb{Z}/2\mathbb{Z}))$ . We call  $\Gamma_2(n)$  the *level 2 principal congruence subgroup* of  $GL(n; \mathbb{Z})$ . McCarthy-Pinkall [11] showed that  $\ker \Phi_g$  is isomorphic to  $\Gamma_2(g-1)$ . On the other hand, Szepietowski [18] gave a finite generating set for  $\Gamma_2(N_g)$ , and then the first author and Sato [6] gave a minimal generating set for  $\Gamma_2(N_g)$ . Fullarton [4], the second author [8], Margalit and Putman gave a finite presentation for  $\Gamma_2(n)$  independently. Therefore, we obtain a normal generating set for  $\mathcal{I}(N_g)$  in  $\Gamma_2(N_g)$ .

Here is an outline of this paper. In Section 2, we will explain about basics on the Torelli group of a non-orientable surface. In Section 3, we will explain about the finite generating set for  $\Gamma_2(N_g)$ , the finite presentation for  $\Gamma_2(n)$  and an isomorphism from  $\ker \Phi_g$  to  $\Gamma_2(g-1)$ . In Section 4, we will obtain a normal generating set for  $\mathcal{I}(N_g)$ . In Section 5, we will show that each normal generator of  $\mathcal{I}(N_g)$  obtained in Section 4 is described as a product of BSCC maps and BP maps.

## 2. BASICS ON THE TORELLI GROUP OF A NON-ORIENTABLE SURFACE.

There are BSCC maps of two types. A BSCC map  $t_c$  is called a BSCC map of type  $(1, h)$  if each connected component of  $N_g \setminus c$  is non-orientable and one component of  $N_g \setminus c$  is homeomorphic to  $N_h^1$  for  $1 \leq h \leq \frac{g}{2}$  (see Figure 5 (a)). A BSCC map  $t_c$  is called a BSCC map of type  $(2, h)$  if one component of  $N_g \setminus c$  is homeomorphic to  $\Sigma_h^1$  for  $1 \leq h < \frac{g}{2}$ , and the other component is non-orientable (see Figure 7). Note that a BSCC map  $t_c$  is trivial if  $c$  bounds a Möbius band (see Theorem 3.4 of [3]).

There are BP maps of two types. A BP map  $t_{c_1}t_{c_2}^{-1}$  is called a BP map of type  $(1, h)$  if one component of  $N_g \setminus (c_1 \cup c_2)$  is homeomorphic to  $\Sigma_h^2$  for  $1 \leq h < \frac{g}{2} - 1$ , and the other component is non-orientable (see Figure 5 (b)). A BP map  $t_{c_1}t_{c_2}^{-1}$  is called a BP map of type  $(2, h)$  if each component of  $N_g \setminus (c_1 \cup c_2)$  is orientable and one component of  $N_g \setminus (c_1 \cup c_2)$  is homeomorphic to  $\Sigma_h^2$  for  $1 \leq h \leq \frac{g}{2} - 1$  (see Figure 7). Note that a BP map of type  $(2, h)$  appears only if  $g$  is even.

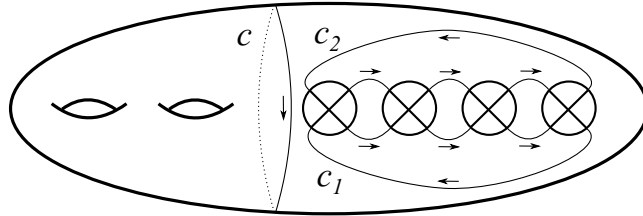


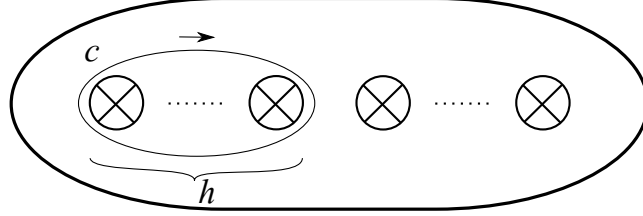
FIGURE 7. A BSCC map  $t_c$  of type  $(2, 2)$  and a BP map  $t_{c_1}t_{c_2}^{-1}$  of type  $(2, 1)$ .

At first, we show the following.

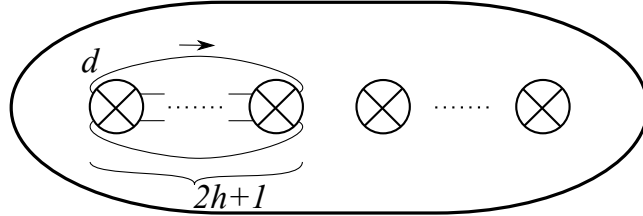
**Remark 2.1.** *All BSCC maps and BP maps are in  $\mathcal{I}(N_g)$ .*

*Proof.* Let  $c, d, c_1, c_2, d_1$  and  $d_2$  be simple closed curves on  $N_g$  as shown in Figure 8. Note that  $t_c$  is a BSCC map of type  $(1, h)$ ,  $t_d$  is a BSCC map of type  $(2, h)$ ,  $t_{c_1}t_{c_2}^{-1}$  is a BP map of type  $(1, h)$  and  $t_{d_1}t_{d_2}^{-1}$  is a BP map of type  $(2, h)$ . In  $\mathcal{M}(N_g)$ , any BSCC map of type

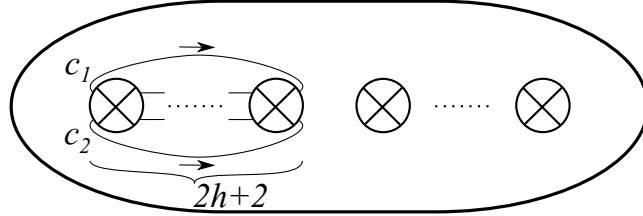
$(1, h)$  (resp. type  $(2, h)$ ) is conjugate to  $t_c^{\pm 1}$  (resp.  $t_d^{\pm 1}$ ), and any BP map of type  $(1, h)$  (resp. type  $(2, h)$ ) is conjugate to  $(t_{c_1} t_{c_2}^{-1})^{\pm 1}$  (resp.  $(t_{d_1} t_{d_2}^{-1})^{\pm 1}$ ). Hence it suffice to show that  $t_c, t_d, t_{c_1} t_{c_2}^{-1}$  and  $t_{d_1} t_{d_2}^{-1}$  are in  $\mathcal{I}(N_g)$ .



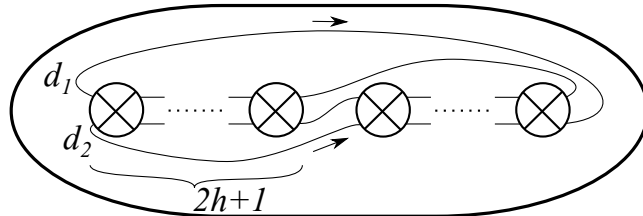
(a) A BSCC map  $t_c$  of type  $(1, h)$ .



(b) A BSCC map  $t_d$  of type  $(2, h)$ .



(c) A BP map  $t_{c_1} t_{c_2}^{-1}$  of type  $(1, h)$ .



(d) A BP map  $t_{d_1} t_{d_2}^{-1}$  of type  $(2, h)$ .

FIGURE 8.

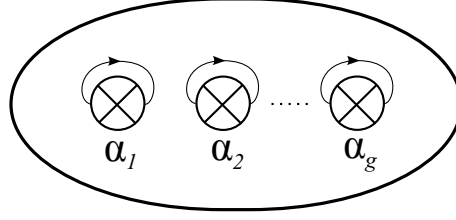
For  $1 \leq i \leq g$ , let  $\alpha_i$  be a simple closed curve on  $N_g$  as shown in Figure 9, and let  $c_i = [\alpha_i] \in H_1(N_g; \mathbb{Z})$ . By a natural handle decomposition whose cores of the 1-handles are  $\alpha_i$ , we have that  $H_1(N_g; \mathbb{Z})$  is generated by  $c_i$ , as a  $\mathbb{Z}$ -module (see Figure 13). We can see that  $t_c, t_d, t_{c_1} t_{c_2}^{-1}$  and  $t_{d_1} t_{d_2}^{-1}$  act trivially on each  $c_i$ .

□

Next we prove the following.

**Lemma 2.2.** *For  $g \geq 5$ , we have followings.*

- (1) (a) Any BSCC map of type  $(2, \frac{g}{2} - 1)$  in the case where  $g$  is even is a product of BP maps of type  $(1, 1)$ .  
 (b) Any BSCC map of the other types is a product of BSCC maps of type  $(1, 2)$ .
- (2) Any BP map of type  $(1, h)$  is a product of BP maps of type  $(1, 1)$ . Any BP map of type  $(2, h)$  is a product of BP maps of type  $(2, 1)$ .

FIGURE 9. The loops  $\alpha_1, \alpha_2, \dots, \alpha_g$ .

*Proof.* In a proof, we use ideas of Johnson [7].

- (1) (a) We first show that a BSCC map of type  $(2, \frac{g}{2} - 1)$  is a product of BP maps. Let  $t_c$  be a BSCC map of type  $(2, \frac{g}{2} - 1)$ . Then the curve  $c$  is as shown in Figure 10 (a). Let  $x, y, z, a, b$  and  $d$  be simple closed curves as shown in Figure 10 (a). By the lantern relation, we have the relation  $t_d t_c t_b t_a = t_z t_y t_x$ . Since  $a, b, c$  and  $d$  are not intersect other loops, we have  $t_c = (t_z t_a^{-1})(t_y t_d^{-1})(t_x t_b^{-1})$ . Note that  $t_x t_b^{-1}$ ,  $t_y t_d^{-1}$  and  $t_z t_a^{-1}$  are BP maps. Hence a BSCC map of type  $(2, \frac{g}{2} - 1)$  is a product of BP maps. As we will show in the assertion (2), these BP maps are products of BP maps of type  $(1, 1)$ . Hence we obtain the claim.
- (b) Let  $t_c$  be a BSCC map of type  $(1, h)$  or  $(2, \frac{g-h}{2})$  for  $h \geq 3$ , then  $c$  is as shown in Figure 10 (b). let  $c_{i,j}$  be a simple closed curve for  $1 \leq i < j \leq h$  as shown in Figure 10 (b). We have

$$(I) \quad t_c = \prod_{1 \leq i \leq h-1} (t_{c_{i,i+1}} t_{c_{i,i+2}} \cdots t_{c_{i,h-1}} t_{c_{i,h}}).$$

The equation (I) will be shown in Appendix A. Since each  $t_{c_{i,j}}$  is a BSCC map of type  $(1, 1)$ , we obtain the claim.

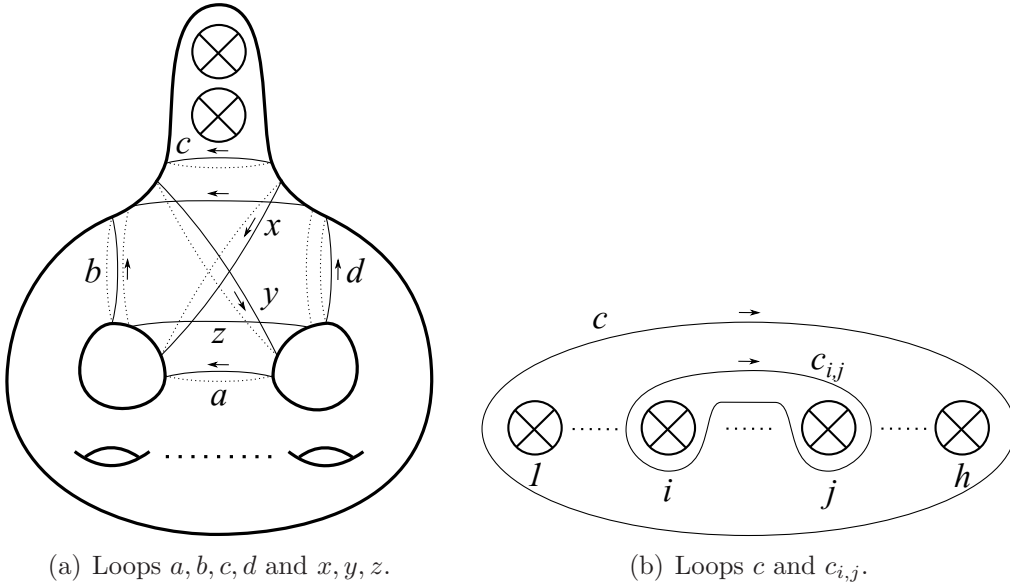


FIGURE 10.

- (2) For  $h \geq 1$ , let  $d_0, d_1, \dots, d_h$  be simple closed curves as shown in Figure 11. Suppose that  $d_0$  and  $d_h$  are not separating curves. Note that  $t_{d_0} t_{d_h}^{-1}$  is a BP map of type

$(1, h)$  or  $(2, h)$ . Then we have the equation  $t_{d_0} t_{d_h}^{-1} = (t_{d_0} t_{d_1}^{-1})(t_{d_1} t_{d_2}^{-1}) \cdots (t_{d_{h-1}} t_{d_h}^{-1})$ . Since each  $t_{d_i} t_{d_{i+1}}^{-1}$  is a BP map of type  $(1, 1)$  or  $(2, 1)$ , we obtain the claim.

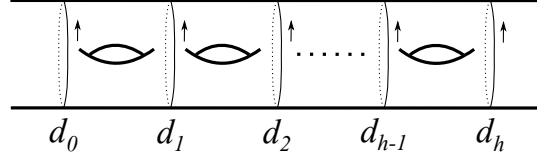


FIGURE 11. Loops  $d_0, d_1, \dots, d_h$ .

□

### 3. PRELIMINARIES

#### 3.1. On generators for $\Gamma_2(N_g)$ .

McCarthy, Pinkall [11] and Gadgil, Pancholi [5] proved that the natural homomorphism  $\rho_2 : \mathcal{M}(N_g) \rightarrow \text{Aut}(H_1(N_g; \mathbb{Z}/2\mathbb{Z}), \cdot)$  is surjective. Szepietowski [17] proved that  $\Gamma_2(N_g)$  is generated by  $Y$ -homeomorphisms, and that  $\Gamma_2(N_g)$  is generated by involutions. Therefore,  $H_1(\Gamma_2(N_g); \mathbb{Z})$  is a  $\mathbb{Z}/2\mathbb{Z}$ -module. The first author and Sato [6] showed that  $H_1(\Gamma_2(N_g); \mathbb{Z})$  is the  $\mathbb{Z}/2\mathbb{Z}$ -module of the rank  $\binom{g}{3} + \binom{g}{2}$ .

For  $I = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, g\}$ , we define an oriented simple closed curve  $\alpha_I$  as shown in Figure 12. For short, we denote  $\alpha_{\{i\}}$  by  $\alpha_i$ . We define  $Y_{i_1; i_2, \dots, i_k} = Y_{\alpha_{i_1}, \alpha_I}$ ,  $T_{i_1, \dots, i_k} = t_{\alpha_I}$  if  $k$  is even.

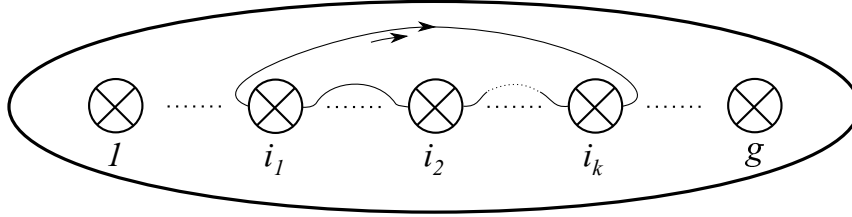


FIGURE 12. The curve  $\alpha_I$  for  $I = \{i_1, i_2, \dots, i_k\}$ .

Szepietowski [18] gave a generating set for  $\Gamma_2(N_g)$  as follows.

**Theorem 3.1** ([18]). *For  $g \geq 4$ ,  $\Gamma_2(N_g)$  is generated by the following elements.*

- (1)  $Y_{i;j}$  for  $1 \leq i \leq g-1$ ,  $1 \leq j \leq g$  and  $i \neq j$ ,
- (2)  $T_{i,j,k,l}^2$  for  $1 \leq i < j < k < l \leq g$ .

In addition, the first author and Sato [6] gave a minimal generating set for  $\Gamma_2(N_g)$  as follows.

**Theorem 3.2** ([6]). *For  $g \geq 4$ ,  $\Gamma_2(N_g)$  is generated by the following elements.*

- (1)  $Y_{i;j}$  for  $1 \leq i \leq g-1$ ,  $1 \leq j \leq g$  and  $i \neq j$ ,
- (2)  $T_{1,j,k,l}^2$  for  $1 < j < k < l \leq g$ .

#### 3.2. On $\ker \Phi_g$ and $\Gamma_2(g-1)$ .

McCarthy and Pinkall claimed that  $\ker \Phi_g$  is isomorphic to  $\Gamma_2(g-1)$  in their unpublished preprint [11]. In this subsection, we refer their result and proof.



Let  $c_i = [\alpha_i] \in H_1(N_g; \mathbb{Z})$  for  $1 \leq i \leq g$ , and let  $c = c_1 + c_2 + \cdots + c_g (= [\alpha_{\{1, \dots, g\}}])$ . Then, by a natural handle decomposition as shown in Figure 13, as a  $\mathbb{Z}$ -module,  $H_1(N_g; \mathbb{Z})$  has a presentation

$$H_1(N_g; \mathbb{Z}) = \langle c_1, c_2, \dots, c_g \mid 2c = 0 \rangle.$$

As a  $\mathbb{Z}$ -module we have

$$\begin{aligned} H_1(N_g; \mathbb{Z}) / \langle c \rangle &= \langle c_1, c_2, \dots, c_g \mid c = 0 \rangle \\ &= \langle c_1, c_2, \dots, c_{g-1} \rangle \\ &\cong \mathbb{Z}^{g-1}, \end{aligned}$$

where we settle that the last isomorphism sends  $c_i$  to the  $i$ -th canonical normal vector  $e_i$  for  $1 \leq i \leq g-1$ . For  $x \in H_1(N_g; \mathbb{Z})$ , we denote by  $\bar{x}$  the image of  $x$  by the projection  $H_1(N_g; \mathbb{Z}) \rightarrow \mathbb{Z}^{g-1}$ . Explicitly, for  $x = \sum_{j=1}^g x_j c_j \in H_1(N_g; \mathbb{Z})$ , we have  $\bar{x} = \sum_{j=1}^{g-1} (x_j - x_g) e_j \in \mathbb{Z}^{g-1}$ . We regard  $\text{Aut}(H_1(N_g; \mathbb{Z}) / \langle c \rangle)$  as  $GL(g-1; \mathbb{Z})$ .

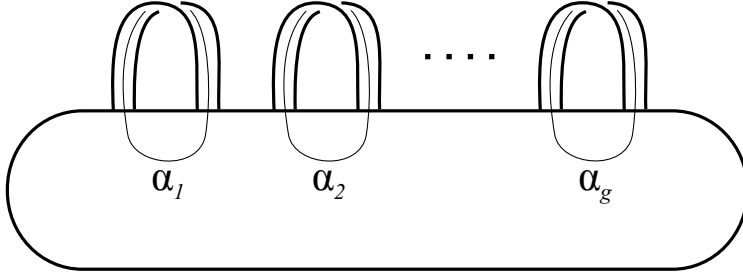


FIGURE 13. A handle decomposition of  $N_g$  whose cores of the 1-handles are  $\alpha_i$ .

For  $L \in \text{Aut}(H_1(N_g; \mathbb{Z}))$ , since  $2L(c) = L(2c) = 0$  and  $c$  is the only non-trivial element of  $H_1(N_g; \mathbb{Z})$  satisfying  $2c = 0$ , we have  $L(c) = c$ . Hence  $L \in \text{Aut}(H_1(N_g; \mathbb{Z}))$  induces  $\bar{L} \in GL(g-1; \mathbb{Z})$ . More precisely,  $\bar{L}$  is defined as  $\bar{L}(e_i) = \overline{L(c_i)}$ . By this correspondence, we obtain the following.

**Proposition 3.3.** *The correspondence  $f : \ker \Phi_g \rightarrow \Gamma_2(g-1)$  defined by  $f(L) = \bar{L}$  is an isomorphism.*

*Proof.* We first show that  $f(\ker \Phi_g)$  is in  $\Gamma_2(g-1)$  and  $f$  is a homomorphism. By the definition of  $\Phi_g$ , we have that  $L(c_i) \equiv c_i \pmod{2}$  for  $1 \leq i \leq g$ . Hence  $\bar{L}(e_i) = \overline{L(c_i)} \equiv \overline{c_i} = e_i \pmod{2}$  for  $1 \leq i \leq g-1$ . Therefore we have  $f(L) \in \Gamma_2(g-1)$ . In addition, for  $L, L' \in \ker \Phi_g$ , we see

$$\begin{aligned} \overline{LL'}(e_i) &= \overline{LL'(c_i)} \\ &= \overline{L(L'(c_i))} \\ &= \overline{L(\overline{L'(c_i)})} \\ &= \overline{L(\bar{L}'(e_i))}. \end{aligned}$$

Thus,  $f$  is a homomorphism.

We next show the injectivity of  $f$ . For  $L \in \ker \Phi_g$ , suppose that  $\bar{L}$  is the identity. Then we have either  $L(c_i) = c_i$  or  $L(c_i) = c_i + c$ . By the definition of  $\Phi_g$ , we have that  $L(c_i) \equiv c_i \pmod{2}$  for  $1 \leq i \leq g$ . Hence  $L$  is the identity. Therefore  $f$  is injective.



Finally we show the surjectivity of  $f$ . For any  $A = (a_{ij}) \in \Gamma_2(g-1)$ , we define  $\tilde{A} \in \ker \Phi_g$  to be

$$\tilde{A}(c_i) = \begin{cases} \sum_{j=1}^{g-1} a_{ji} c_j & (i \neq g), \\ \sum_{j=1}^{g-1} (1 - \sum_{k=1}^{g-1} a_{jk}) c_j + c_g & (i = g). \end{cases}$$

Then we have that  $\tilde{A}(c) = c$  and, since  $a_{ii}$  is odd and  $a_{ij}$  is even for  $i \neq j$ ,  $\tilde{A}(c_i) \equiv c_i \pmod{2}$ . Hence we have  $\tilde{A} \in \ker \Phi_g$ . In addition, we have  $f(\tilde{A}) = A$ . Therefore  $f$  is surjective.

Thus  $f$  is an isomorphism.  $\square$

### 3.3. On a presentation for $\Gamma_2(g-1)$ .

For  $1 \leq i, j \leq n$  with  $i \neq j$ , let  $E_{ij}$  denote the matrix whose  $(i, j)$  entry is 2, diagonal entries are 1 and others are 0, and let  $F_i$  denote the matrix whose  $(i, i)$  entry is  $-1$ , other diagonal entries are 1 and others are 0. It is known that  $\Gamma_2(n)$  is generated by  $E_{ij}$  and  $F_i$  (see [11]). In addition, a finite presentation for  $\Gamma_2(n)$  was independently given by Fullarton [4], the second author [8], Margalit and Putman recently.

**Theorem 3.4** (cf. [4], [8]). *For  $n \geq 1$ ,  $\Gamma_2(n)$  has a finite presentation with generators  $E_{ij}$  and  $F_i$ , for  $1 \leq i, j \leq n$ , and with relators*

- (1)  $F_i^2$ ,
- (2)  $(E_{ij}F_i)^2, (E_{ij}F_j)^2, (F_iF_j)^2$  (when  $n \geq 2$ ),
- (3) (a)  $[E_{ij}, E_{ik}], [E_{ij}, E_{kj}], [E_{ij}, F_k], [E_{ij}, E_{ki}]E_{kj}^2$  (when  $n \geq 3$ ),  
 (b)  $(E_{ji}E_{ij}^{-1}E_{kj}^{-1}E_{jk}E_{ik}E_{ki}^{-1})^2$  for  $i < j < k$  (when  $n \geq 3$ ),
- (4)  $[E_{ij}, E_{kl}]$  (when  $n \geq 4$ ),

where  $[X, Y] = X^{-1}Y^{-1}XY$  and  $1 \leq i, j, k, l \leq n$  are mutually different.

For  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$  with  $i \neq j$ , let  $Y_{ij} = f((Y_{i,j})_*)$ , where  $\varphi_* \in \text{Aut}(H_1(N_g; \mathbb{Z}), \cdot)$  means the automorphism induced by  $\varphi \in \mathcal{M}(N_g)$ . Then we have that  $Y_{ij} = E_{ij}F_i$  if  $j < g$  and  $Y_{ig} = F_i$ . We now prove the following.

**Proposition 3.5.** *For  $g-1 \geq 1$ ,  $\Gamma_2(g-1)$  has a finite presentation with generators  $Y_{ij}$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$  with  $i \neq j$ , and with relators*

- (1)  $Y_{ij}^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$ ,
- (2)  $[Y_{ik}, Y_{jk}]$  for  $1 \leq i, j \leq g-1$  and  $1 \leq k \leq g$ ,
- (3)  $[Y_{ij}, Y_{ik}Y_{jk}]$  for  $1 \leq i, j \leq g-1$  and  $1 \leq k \leq g$ ,
- (4)  $[Y_{ij}, Y_{kl}]$  for  $1 \leq i, k \leq g-1$  and  $1 \leq j, l \leq g$ ,
- (5)  $(Y_{ij}Y_{ik}Y_{il})^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k, l \leq g$ ,
- (6)  $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$  for  $1 \leq i, j, k \leq g-1$ ,

where  $[X, Y] = X^{-1}Y^{-1}XY$  and  $i, j, k, l$  are mutually different.

*Proof.* At first, we show that the relators of  $\Gamma_2(g-1)$  for Proposition 3.5 are obtained from that for Theorem 3.4. By the relation  $F_i^2 = 1$ , we may identify  $F_i^{-1}$  with  $F_i$  in  $\Gamma_2(g-1)$ .

- (1) We have  $Y_{ij}^2 = \begin{cases} (E_{ij}F_i)^2 & (j < g), \\ F_i^2 & (j = g). \end{cases}$  Hence we obtain the relator  $Y_{ij}^2$  in  $\Gamma_2(g-1)$ .

(2) For  $k < g$ , we see

$$\begin{aligned}
Y_{ik}Y_{jk} &= \frac{E_{ik}F_iE_{jk}F_j}{(3)(a)} \\
&= \frac{E_{jk}E_{ik}F_iF_j}{(2), (3)(a)} \\
&= E_{jk}F_jE_{ik}F_i \\
&= Y_{jk}Y_{ik}.
\end{aligned}$$

In addition, we see

$$\begin{aligned}
Y_{ig}Y_{jg} &= F_iF_j \\
&\stackrel{(2)}{=} F_jF_i \\
&= Y_{jg}Y_{ig}.
\end{aligned}$$

Hence we obtain the relator  $[Y_{ik}, Y_{jk}]$  in  $\Gamma_2(g-1)$ .

(3) For  $k < g$ , we see

$$\begin{aligned}
Y_{ij}Y_{ik}Y_{jk} &= \frac{E_{ij}F_iE_{ik}F_iE_{jk}F_j}{(2) (3)(a), (2)} \\
&= \frac{E_{ij}E_{ik}^{-1}F_iE_{jk}F_jF_i}{(3)(a), (2)} \\
&= \frac{E_{ik}^{-1}F_iE_{ij}^{-1}E_{jk}F_jF_i}{(3)(a)} \\
&= \frac{E_{ik}^{-1}F_iE_{ik}^{-2}E_{jk}E_{ij}^{-1}F_jF_i}{(2) (2)} \\
&= E_{ik}F_iE_{jk}F_jE_{ij}F_i \\
&= Y_{ik}Y_{jk}Y_{ij}.
\end{aligned}$$

In addition, we see

$$\begin{aligned}
Y_{ij}Y_{ig}Y_{jg} &= \frac{E_{ij}F_i}{(2)} \frac{F_iF_j}{(2)} \\
&= \frac{F_iE_{ij}^{-1}F_jF_i}{(2)} \\
&= F_iF_jE_{ij}F_i \\
&= Y_{ig}Y_{jg}Y_{ij}.
\end{aligned}$$

Hence we obtain the relator  $[Y_{ij}, Y_{ik}Y_{jk}]$  in  $\Gamma_2(g-1)$ .

(4) For  $j, l < g$ , we see

$$\begin{aligned}
Y_{ij}Y_{kl} &= \frac{E_{ij}F_iE_{kl}F_k}{(3)(a), (2)} \\
&= \frac{E_{ij}E_{kl}F_kF_i}{(4), (3)(a)} \\
&= E_{kl}F_kE_{ij}F_i \\
&= Y_{kl}Y_{ij}.
\end{aligned}$$

In addition, we see

$$\begin{aligned} Y_{ij}Y_{kg} &= E_{ij}F_iF_k \\ &\stackrel{(2), (3)(a)}{=} F_kE_{ij}F_i \\ &= Y_{kg}Y_{ij}. \end{aligned}$$

Hence we obtain the relator  $[Y_{ij}, Y_{kl}]$  in  $\Gamma_2(g-1)$ .

- (5) For the relator  $(Y_{ij}Y_{ik}Y_{il})^2$ , by the fact  $Y_{im} = Y_{im}^{-1}$ , applying conjugations and taking their inverses, it suffices to consider the case  $j < k < l$ . For  $l < g$ , we see

$$\begin{aligned} Y_{ij}Y_{ik}Y_{il} &= \frac{E_{ij}F_iE_{ik}}{(2)} \frac{F_iE_{il}F_i}{(2)} \\ &= \frac{E_{ij}E_{ik}^{-1}F_iE_{il}^{-1}F_iF_i}{(3)(a), (2)} \\ &= \frac{E_{ik}^{-1}F_iE_{ij}^{-1}E_{il}^{-1}F_iF_i}{(3)(a), (2)} \\ &= \frac{E_{ik}^{-1}F_iE_{il}^{-1}F_iE_{ij}F_i}{(2)} \\ &= \frac{E_{ik}^{-1}E_{il}F_iF_iE_{ij}F_i}{(3)(a), (2)} \\ &= E_{il}F_iE_{ik}F_iE_{ij}F_i \\ &= Y_{il}Y_{ik}Y_{ij} \\ &= Y_{il}^{-1}Y_{ik}^{-1}Y_{ij}^{-1}. \end{aligned}$$

In addition, we see

$$\begin{aligned} Y_{ij}Y_{ik}Y_{ig} &= \frac{E_{ij}F_iE_{ik}F_iF_i}{(2), (3)(a)} \\ &= F_iE_{ik}F_iE_{ij}F_i \\ &= Y_{ig}Y_{ik}Y_{ij} \\ &= Y_{ig}^{-1}Y_{ik}^{-1}Y_{ij}^{-1}. \end{aligned}$$

Hence we obtain the relator  $(Y_{ij}Y_{ik}Y_{il})^2$  in  $\Gamma_2(g-1)$ .

- (6) We see

$$\begin{aligned} Y_{ji}Y_{ij} \cdot Y_{kj}Y_{jk} \cdot Y_{ik}Y_{ki} &= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}F_iE_{ki}F_k) \\ &\stackrel{(2)}{=} (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}\underline{F_j})(E_{ik}E_{ki}^{-1})F_iF_k \\ &\stackrel{(3)(a)}{=} (E_{ji}F_jE_{ij}F_i)(E_{kj}\underline{F_k}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_k \\ &\stackrel{(2)}{=} (E_{ji}F_jE_{ij}\underline{F_i})(E_{kj}E_{jk}^{-1})(E_{ik}^{-1}E_{ki})F_kF_jF_iF_k \\ &\stackrel{(3)(a), (2)}{=} (E_{ji}\underline{F_j}E_{ij})(E_{kj}E_{jk}^{-1})(E_{ik}E_{ki}^{-1})F_iF_kF_jF_iF_k \\ &\stackrel{(2), (3)(a)}{=} (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_kF_jF_iF_k \\ &\stackrel{(1), (2)}{=} (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1}). \end{aligned}$$

By (3) (b) of Theorem 3.4, we obtain the relator  $(Y_{ji}Y_{ij} \cdot Y_{kj}Y_{jk} \cdot Y_{ik}Y_{ki})^2$  in  $\Gamma_2(g-1)$ .

Next, we show that the relators of  $\Gamma_2(g-1)$  for Theorem 3.4 are obtained from that for Proposition 3.5. By the relation  $Y_{ij}^2 = 1$ , we may identify  $Y_{ij}^{-1}$  with  $Y_{ij}$  in  $\Gamma_2(g-1)$ . Note that  $E_{ij} = Y_{ij}Y_{ig}$  and  $F_i = Y_{ig}$ .

- (1) Since  $F_i^2 = Y_{ig}^2$ , we have the relator  $F_i^2$  in  $\Gamma_2(g-1)$ .
- (2) Since  $E_{ij}F_i = Y_{ij}$ , we have the relator  $(E_{ij}E_i)^2$  in  $\Gamma_2(g-1)$ . We see

$$\begin{aligned}
 (E_{ij}F_j)^2 &= \frac{Y_{ij}}{(1)} \frac{Y_{ig}Y_{jg}}{(1), (2)} \cdot Y_{ij}Y_{ig}Y_{jg} \\
 &= Y_{ij}^{-1}(Y_{ig}Y_{jg})^{-1} \cdot Y_{ij}Y_{ig}Y_{jg} \\
 &= [Y_{ij}, Y_{ig}Y_{jg}].
 \end{aligned}$$

Hence we obtain the relator  $(E_{ij}F_j)^2$  in  $\Gamma_2(g-1)$ . We see

$$\begin{aligned}
 (F_iF_j)^2 &= \frac{Y_{ig}Y_{jg}Y_{ig}Y_{jg}}{(2)} \\
 &= Y_{ig}^2Y_{jg}^2.
 \end{aligned}$$

Hence we obtain the relator  $(F_iF_j)^2$  in  $\Gamma_2(g-1)$ .

- (3) (a) We see

$$\begin{aligned}
 E_{ij}E_{ik} &= \frac{Y_{ij}Y_{ig}Y_{ik}Y_{ig}}{(5)} \\
 &= Y_{ik}Y_{ig}Y_{ij}Y_{ig} \\
 &= E_{ik}E_{ij}.
 \end{aligned}$$

Hence we obtain the relator  $[E_{ij}, E_{ik}]$  in  $\Gamma_2(g-1)$ . We see

$$\begin{aligned}
 E_{ij}E_{kj} &= \frac{Y_{ij}Y_{ig}Y_{kj}Y_{kg}}{(4), (2)} \\
 &= \frac{Y_{ij}Y_{kj}Y_{kg}Y_{ig}}{(2), (4)} \\
 &= Y_{kj}Y_{kg}Y_{ij}Y_{ig} \\
 &= E_{kj}E_{ij}.
 \end{aligned}$$

Hence we obtain the relator  $[E_{ij}, E_{kj}]$  in  $\Gamma_2(g-1)$ . We see

$$\begin{aligned}
 E_{ij}F_k &= Y_{ij}Y_{ig}Y_{kg} \\
 &= \frac{Y_{kg}Y_{ij}Y_{ig}}{(2), (4)} \\
 &= F_kE_{ij}.
 \end{aligned}$$

Hence we obtain the relator  $[E_{ij}, F_k]$  in  $\Gamma_2(g-1)$ . We see

$$\begin{aligned}
E_{ij}E_{ki}E_{kj}^2 &= Y_{ij}Y_{ig}\underbrace{Y_{ki}Y_{kg}Y_{kj}Y_{kg}Y_{kj}Y_{kg}}_{(5)} \\
&= Y_{ij}\underbrace{Y_{ig}Y_{kj}Y_{kg}Y_{ki}Y_{kg}Y_{kj}Y_{kg}}_{(4), (2)} \\
&= \underbrace{Y_{ij}Y_{kj}}_{(2)}\underbrace{Y_{kg}Y_{ig}Y_{ki}Y_{kg}Y_{kj}Y_{kg}}_{(3)} \\
&= \underbrace{Y_{kj}Y_{ij}Y_{ki}Y_{kg}}_{(3)}\underbrace{Y_{ig}Y_{kg}Y_{kj}Y_{kg}}_{(2), (4)} \\
&= Y_{ki}Y_{kj}Y_{ij}\underbrace{Y_{kg}Y_{kg}Y_{kj}Y_{kg}Y_{ig}}_{(1)} \\
&= Y_{ki}Y_{kj}\underbrace{Y_{ij}Y_{kj}Y_{kg}Y_{ig}}_{(2), (4)} \\
&= Y_{ki}\underbrace{Y_{kj}Y_{kj}Y_{kg}Y_{ig}}_{(1)} \\
&= Y_{ki}Y_{kg}Y_{ij}Y_{ig} \\
&= E_{ki}E_{ij}.
\end{aligned}$$

Hence we obtain the relator  $[E_{ij}, E_{ki}]E_{kj}^2$  in  $\Gamma_2(g-1)$ .

- (b) Since we already obtained relators (1), (2) and (a) of (3) for Theorem 3.4, using these relators, we see

$$\begin{aligned}
(E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1}) &= (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_kF_jF_iF_k \\
&= (E_{ji}F_jE_{ij})(E_{kj}E_{jk}^{-1})(E_{ik}E_{ki}^{-1})F_iF_kF_jF_iF_k \\
&= (E_{ji}F_jE_{ij}F_i)(E_{kj}E_{jk}^{-1})(E_{ik}^{-1}E_{ki})F_kF_jF_iF_k \\
&= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_k \\
&= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}E_{ki}^{-1})F_iF_k \\
&= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}F_iE_{ki}F_k) \\
&= Y_{ji}Y_{ij} \cdot Y_{kj}Y_{jk} \cdot Y_{ik}Y_{ki}.
\end{aligned}$$

By Proposition 3.5 (6), we obtain the relator  $(E_{ji}E_{ij}^{-1}E_{kj}^{-1}E_{jk}E_{ik}E_{ki}^{-1})^2$  in  $\Gamma_2(g-1)$ .

(4) We see

$$\begin{aligned}
E_{ij}E_{kl} &= Y_{ij}\underbrace{Y_{ig}Y_{kl}Y_{kg}}_{(2), (4)} \\
&= \underbrace{Y_{ij}Y_{kl}Y_{kg}Y_{ig}}_{(4)} \\
&= Y_{kl}Y_{kg}Y_{ij}Y_{ig} \\
&= E_{kl}E_{ij}.
\end{aligned}$$

Hence we obtain the relator  $[E_{ij}, E_{kl}]$  in  $\Gamma_2(g-1)$ .

Thus, we complete the proof.  $\square$

#### 4. A NORMAL GENERATING SET FOR $\mathcal{I}(N_g)$

Let  $f : \ker \Phi_g \rightarrow \Gamma_2(g-1)$  be the isomorphism introduced in Subsection 3.2. In order to obtain a presentation for  $\Gamma_2(g-1)$  whose generators are  $Y_{ij} := f((Y_{ij})_*)$ ,  $T_{1jkl}^2 := f((T_{1,j,k,l}^2)_*)$ , we need to express  $T_{1jkl}^2$  as a product of  $Y_{ij}$ 's.

For  $I = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, g\}$ , we define a simple closed curve  $\alpha'_I$  as shown in Figure 14 and  $T'_{i,j,k,l} = t_{\alpha'_{\{i,j,k,l\}}}$ . Note that  $T_{i,j,k,l} T'^{-1}_{i,j,k,l}$  is a BP map. In addition, for  $1 \leq m \leq g$  with  $m \neq i, j, k, l$ , there exist  $A$ -circles  $\beta_{m,i}$ ,  $\beta_{m,j}$ ,  $\beta_{m,k}$  and  $\beta_{m,l}$  intersecting  $\alpha_m$  at only one point such that

$$T_{i,j,k,l}^{-1} T'^{-1}_{i,j,k,l} = \prod_{m \neq i,j,k,l} Y_{\alpha_m, \beta_{m,l}} Y_{\alpha_m, \beta_{m,k}} Y_{\alpha_m, \beta_{m,j}} Y_{\alpha_m, \beta_{m,i}}.$$

For example, when  $(i, j, k, l)$  is  $(1, 2, 3, 4)$ , for  $m \geq 5$  and  $t = 1, 2, 3, 4$  we have

$$Y_{\alpha_m, \beta_{m,t}} = \begin{cases} Y_{m;t}^{-1} & (t = 1, 2, 3), \\ Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2} Y_{m;5}^{-2} Y_{m;4}^{-1} & (t = 4). \end{cases}$$

Therefore, we have

$$\begin{aligned} T_{1jkl}^{-2} &= f((T_{1,j,k,l}^{-1} T'^{-1}_{1,j,k,l})_*) \\ &= f((\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_{m,l}} Y_{\alpha_m, \beta_{m,k}} Y_{\alpha_m, \beta_{m,j}} Y_{\alpha_m, \beta_{m,1}})_*). \end{aligned}$$

Note that any  $Y$ -homeomorphism  $Y_{\alpha_m, \beta}$  is a product of some  $Y_{i;j}$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$  with  $i \neq j$ . For example, we have

$$\begin{aligned} \text{(II)} \quad Y_{g;i} &= (Y_{1;2}^2 \cdots Y_{1;g-1}^2 Y_{1;i}^{-1} Y_{1;g}) \cdots (Y_{i-1;i}^2 \cdots Y_{i-1;g-1}^2 Y_{i-1;i}^{-1} Y_{i-1;g}) \\ &\quad \cdot (Y_{i+1;i+2}^2 \cdots Y_{i+1;g-1}^2 Y_{i+1;i}^{-1} Y_{i+1;g}) \cdots (Y_{g-2;g-1}^2 Y_{g-2;i}^{-1} Y_{g-2;g}) Y_{g-2;i}^2 \\ &\quad \cdot (Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2) Y_{i;g}. \end{aligned}$$

The equation (II) will be shown in Appendix A. Thus  $T_{1jkl}^2$  can be expressed as a product of  $Y_{ij}$ 's.

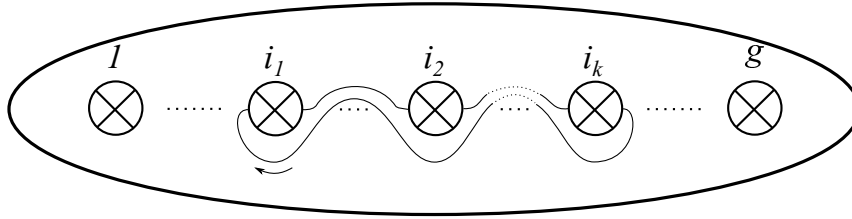


FIGURE 14. The curve  $\alpha'_I$  for  $I = \{i_1, i_2, \dots, i_k\}$ .

##### 4.1. A normal generating set for $\mathcal{I}(N_g)$ in $\Gamma_2(N_g)$ .

Let  $G$  be a group. For  $x_1, x_2, \dots, x_n \in G$ , we say that  $N$  is *normally generated* by  $x_1, x_2, \dots, x_n$  in  $G$  if  $N$  is a minimal normal subgroup of  $G$  which contains  $x_1, x_2, \dots, x_n$ .

In this subsection, we prove the following.

**Proposition 4.1.** *For  $g \geq 4$ ,  $\mathcal{I}(N_g)$  is normally generated by the following elements in  $\Gamma_2(N_g)$ .*

- (1)  $Y_{i;j}^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$ ,
- (2)  $[Y_{i;k}, Y_{j;k}]$  for  $1 \leq i, j \leq g-1$  and  $1 \leq k \leq g$ ,
- (3)  $[Y_{i;j}, Y_{i;k} Y_{j;k}]$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k \leq g$ ,

- (4)  $[Y_{i,j}, Y_{k,l}]$  for  $1 \leq i, k \leq g-1$  and  $1 \leq j, l \leq g$ ,
- (5)  $(Y_{i,j}Y_{i,k}Y_{i,l})^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k, l \leq g$ ,
- (6)  $(Y_{j,i}Y_{i,j}Y_{k,j}Y_{j,k}Y_{i,k}Y_{k,i})^2$  for  $1 \leq i, j, k \leq g-1$ ,
- (7)  $T_{1,j,k,l}^2(\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_m, l} Y_{\alpha_m, \beta_m, k} Y_{\alpha_m, \beta_m, j} Y_{\alpha_m, \beta_m, 1})$  for  $1 < j < k < l \leq g$ ,

where  $i, j, k, l$  are mutually different.

Let  $\Gamma = \langle g_1, g_2, \dots, g_n \mid r_1, r_2, \dots, r_k \rangle$  be a finitely presented group, and let  $G$  be a group generated by  $\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_n$ . For  $r_i = g_{i(1)}^{\varepsilon_1} g_{i(2)}^{\varepsilon_2} \cdots g_{i(m)}^{\varepsilon_m}$ , define  $\tilde{r}_i = \tilde{g}_{i(1)}^{\varepsilon_1} \tilde{g}_{i(2)}^{\varepsilon_2} \cdots \tilde{g}_{i(m)}^{\varepsilon_m}$ , where  $\varepsilon_j = \pm 1$ . Let  $\tilde{N}$  be a normal subgroup of  $G$  normally generated by  $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_k$ . We first prove the following.

**Lemma 4.2.** *Let  $\mu : G \rightarrow \Gamma$  be a homomorphism sending  $\tilde{g}_i$  to  $g_i$ . Then we have  $\tilde{N} = \ker \mu$ .*

*Proof.* Let  $F = \langle g_1, g_2, \dots, g_n \rangle$ , and let  $N$  be a normal subgroup of  $F$  normally generated by  $r_1, r_2, \dots, r_k$ . Let  $\pi : F \rightarrow \Gamma$  be a natural epimorphism, and let  $\nu : F \rightarrow G$  be a homomorphism sending  $g_i$  to  $\tilde{g}_i$ . Since  $\mu$  is surjective, we have  $\pi = \mu\nu$ . Then we have the following short exact sequences and commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{\pi} & \Gamma \longrightarrow 1 \\ & & \downarrow \nu|_N & & \downarrow \nu & & \parallel \\ 1 & \longrightarrow & \ker \mu & \longrightarrow & G & \xrightarrow{\mu} & \Gamma \longrightarrow 1 \end{array}$$

For any  $\tilde{R} \in \ker \mu$ , there exists  $R \in F$  such that  $\nu(R) = \tilde{R}$ . Then we have that  $\pi(R) = \mu\nu(R) = \mu(\tilde{R}) = 1$ . Hence we have  $R \in \ker \pi = N$ . Therefore  $\nu|_N : N \rightarrow \ker \mu$  is surjective. Since  $\nu(N) = \tilde{N}$ , we conclude that  $\tilde{N} = \ker \mu$ .  $\square$

*Proof of Proposition 4.1.* Let  $F$  be a free group of the rank  $\binom{g}{3} + \binom{g}{2}$  generated by  $Y_{ij}$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$  with  $i \neq j$ , and  $T_{1,jkl}^2$  for  $1 < j < k < l \leq g$ , where  $T_{1,jkl} = f((T_{1,j,k,l})_*)$ , and let  $N$  be a normal subgroup of  $F$  normally generated by followings.

- (1)  $Y_{ij}^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$ ,
- (2)  $[Y_{ik}, Y_{jk}]$  for  $1 \leq i, j \leq g-1$  and  $1 \leq k \leq g$ ,
- (3)  $[Y_{ij}, Y_{ik}Y_{jk}]$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k \leq g$ ,
- (4)  $[Y_{ij}, Y_{kl}]$  for  $1 \leq i, k \leq g-1$  and  $1 \leq j, l \leq g$ ,
- (5)  $(Y_{ij}Y_{ik}Y_{il})^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k, l \leq g$ ,
- (6)  $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$  for  $1 \leq i, j, k \leq g-1$ ,
- (7)  $T_{1,jkl}^2 f((\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_m, l} Y_{\alpha_m, \beta_m, k} Y_{\alpha_m, \beta_m, j} Y_{\alpha_m, \beta_m, 1})_*)$  for  $1 < j < k < l \leq g$ ,

where  $i, j, k, l$  are mutually different. By Proposition 3.5 and the fact that

$$f((\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_m, l} Y_{\alpha_m, \beta_m, k} Y_{\alpha_m, \beta_m, j} Y_{\alpha_m, \beta_m, 1})_*) = T_{1,jkl}^{-2},$$

$\Gamma_2(g-1)$  is the quotient of  $F$  by  $N$ . Let  $\tilde{N}$  be the normal subgroup of  $\Gamma_2(N_g)$  normally generated by followings.

- (1)  $Y_{ij}^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j \leq g$ ,
- (2)  $[Y_{i,k}, Y_{j,k}]$  for  $1 \leq i, j \leq g-1$  and  $1 \leq k \leq g$ ,
- (3)  $[Y_{i,j}, Y_{i,k}Y_{j,k}]$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k \leq g$ ,
- (4)  $[Y_{i,j}, Y_{k,l}]$  for  $1 \leq i, k \leq g-1$  and  $1 \leq j, l \leq g$ ,
- (5)  $(Y_{i,j}Y_{i,k}Y_{i,l})^2$  for  $1 \leq i \leq g-1$  and  $1 \leq j, k, l \leq g$ ,



- (6)  $(Y_{j;i}Y_{i;j}Y_{k;j}Y_{j;k}Y_{i;k}Y_{k;i})^2$  for  $1 \leq i, j, k \leq g-1$ ,
- (7)  $T_{1,j,k,l}^2(\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_m, l} Y_{\alpha_m, \beta_m, k} Y_{\alpha_m, \beta_m, j} Y_{\alpha_m, \beta_m, 1})$  for  $1 < j < k < l \leq g$ ,

where  $i, j, k, l$  are mutually different. Let  $\pi : F \rightarrow \Gamma_2(g-1)$  be a natural epimorphism, and let  $\nu : F \rightarrow \Gamma_2(N_g)$  be a homomorphism sending  $Y_{ij}$  to  $Y_{i;j}$ ,  $T_{1jkl}^2$  to  $T_{1,j,k,l}^2$ . Then we have the following short exact sequences and commutative diagram.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{\pi} & \Gamma_2(g-1) \longrightarrow 1 \\
 & & \downarrow \nu|_N & & \downarrow \nu & & \downarrow f^{-1} \\
 1 & \longrightarrow & \ker \rho' & \longrightarrow & \Gamma_2(N_g) & \xrightarrow{\rho'} & \ker \Phi_g \longrightarrow 1
 \end{array}$$

By Lemma 4.2, we have  $\tilde{N} = \ker \rho'$ . On the other hand, since  $\mathcal{I}(N_g) = \ker \rho'$ , we obtain the claim.  $\square$

#### 4.2. A normal generating set for $\mathcal{I}(N_g)$ in $\mathcal{M}(N_g)$ .

In this subsection, we prove the following.

**Proposition 4.3.** *For  $g \geq 4$ ,  $\mathcal{I}(N_g)$  is normally generated by the following elements in  $\mathcal{M}(N_g)$ .*

- (1)  $Y_{1;2}^2$ ,
- (2)  $[Y_{1;3}, Y_{2;3}]$ ,
- (3)  $[Y_{1;2}, Y_{1;3}Y_{2;3}]$ ,  $[Y_{1;3}, Y_{1;2}Y_{3;2}]$ ,
- (4)  $[Y_{1;2}, Y_{3;4}]$ ,  $[Y_{1;3}, Y_{2;4}]$ ,
- (5)  $(Y_{1;2}Y_{1;3}Y_{1;4})^2$ ,
- (6)  $(Y_{2;1}Y_{1;2}Y_{3;2}Y_{2;3}Y_{1;3}Y_{3;1})^2$ ,
- (7)  $T_{1,2,3,4}^2(\prod_{5 \leq m \leq g} Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2} Y_{m;5}^{-2} Y_{m;4}^{-1} Y_{m;3}^{-1} Y_{m;2}^{-1} Y_{m;1}^{-1})$ .

For  $1 \leq i < j \leq g$ , let  $c_{ij}$  be simple closed curve on  $N_g$  as shown in Figure 15, and let  $U_{i,j}$  be a diffeomorphism over  $N_g$  as shown in Figure 15. Note that  $Y_{i;j} = U_{i,j}T_{i,j}$  and  $U_{i,j}^2 = t_{c_{ij}} = Y_{i;j}^2$ .

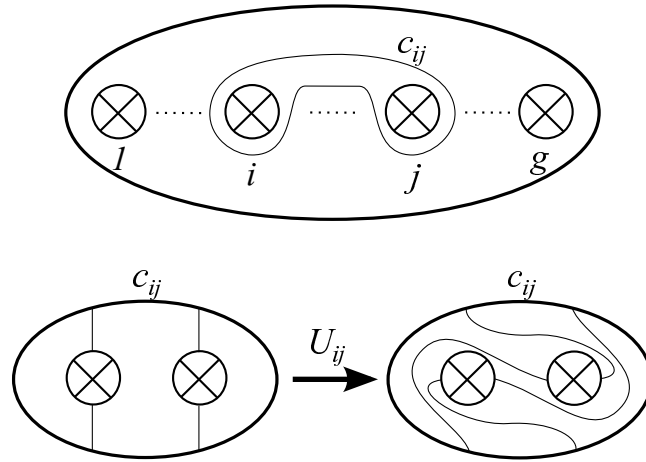


FIGURE 15. The loop  $c_{ij}$  and the diffeomorphism  $U_{ij}$  over  $N_g$ .

Since  $\mathcal{I}(N_g)$  is a normal subgroup of  $\Gamma_2(N_g)$ , the normal generating set for  $\mathcal{I}(N_g)$  in  $\Gamma_2(N_g)$  is a normal generating set for  $\mathcal{I}(N_g)$  in  $\mathcal{M}(N_g)$ . In addition, each normal

generator for  $\mathcal{I}(N_g)$  in Proposition 4.1 is conjugation of some normal generator for  $\mathcal{I}(N_g)$  in Proposition 4.3 by a product of some  $U_{i,j}$ . For example, we have

$$Y_{i,j}^2 = (U_{i-i,i} \cdots U_{1,2})(U_{j-i,j} \cdots U_{2,3})Y_{1,2}^2(U_{2,3}^{-1} \cdots U_{j-1,j}^{-1})(U_{1,2}^{-1} \cdots U_{i-1,i}^{-1})$$

for  $i < j$ . Thus we finish the proof of Proposition 4.3.

## 5. PROOF OF THEOREM 1.1

We put a point  $*$   $\in N_{g-1}$ . Let  $\gamma_1, \gamma_2, \dots, \gamma_{g-1}$  be oriented loops on  $N_{g-1}$  starting at  $*$  as shown in Figure 16. Note that  $\pi_1(N_{g-1}, *)$  is generated by  $[\gamma_1], [\gamma_2], \dots, [\gamma_{g-1}]$ . For  $1 \leq i \leq g$ , let  $s_i : \pi_1(N_{g-1}, *) \rightarrow \mathcal{M}(N_g)$  be the crosscap pushing map defined in [18], such that  $s_i([\gamma_j]) = Y_{i,j}$  if  $j < i$ ,  $s_i([\gamma_j]) = Y_{i,j+1}$  if  $j \geq i$ . We note that the crosscap pushing map is an anti-homomorphism. Namely, we have  $s_i([\alpha][\beta]) = s_i([\beta])s_i([\alpha])$  for  $[\alpha], [\beta] \in \pi_1(N_{g-1}, *)$ . For  $[\alpha] \in \pi_1(N_{g-1}, *)$ ,  $s_i([\alpha])$  is a  $Y$ -homeomorphism if  $\alpha$  is an  $M$ -circle,  $s_i([\alpha])$  is a product of two Dehn twists if  $\alpha$  is an  $A$ -circle.

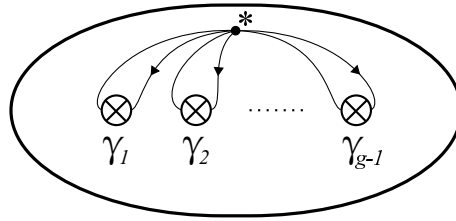


FIGURE 16. The loops  $\gamma_1, \gamma_2, \dots, \gamma_{g-1}$ .

We have the following as a corollary of Proposition 4.3.

**Corollary 5.1.** *For  $g \geq 4$ ,  $\mathcal{I}(N_g)$  is normally generated by the following elements in  $\mathcal{M}(N_g)$ .*

- (1)  $Y_{1,2}^2$ ,
- (2)  $[Y_{2,3}, Y_{1,3}^{-1}]$ ,
- (3)  $[Y_{1,2}, Y_{1,3}Y_{2,3}]$ ,  $[Y_{1,3}, Y_{3,2}Y_{1,2}]$ ,
- (4)  $[Y_{1,2}, Y_{3,4}]$ ,  $[Y_{2,3}^{-2}Y_{1,3}Y_{2,3}^2, Y_{2,4}]$ ,
- (5)  $(Y_{1,2}Y_{1,3}Y_{1,4})^2$ ,
- (6)  $(Y_{2,1}^{-1}Y_{1,2}Y_{3,2}Y_{2,3}^{-1}Y_{1,3}^{-1}Y_{3,1})^2$ ,
- (7)  $T_{1,2,3,4}^2(\prod_{5 \leq m \leq g} Y_{m,m-1}^{-2} \cdots Y_{m,6}^{-2}Y_{m,5}^{-1}Y_{m,4}^{-1}Y_{m,3}^{-1}Y_{m,2}^{-1}Y_{m,1}^{-1})$ .

We now prove Theorem 1.1.

*Proof of Theorem 1.1.* We show that each normal generator for  $\mathcal{I}(N_g)$  in Corollary 5.1 is a product of BSCC maps and BP maps.

- (1) We have that  $Y_{1,2}^2 = s_1([\gamma_1^2]) = t_{c_1}t_{c_2}$ , where  $c_1$  and  $c_2$  are simple closed curves as shown in Figure 17. Note that  $t_{c_1}$  is a BSCC map of type (1, 2). Since  $c_2$  bounds a Möbius band,  $t_{c_2}$  is trivial. Hence we have that  $Y_{1,2}^2$  is a BSCC map of type (1, 2).

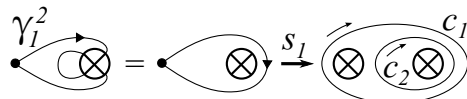


FIGURE 17.

(2) We have that  $Y_{1;3}Y_{2;3}Y_{1;3}^{-1} = Y_{1;3}s_2([\gamma_2])Y_{1;3}^{-1} = s_2([\gamma_1^2\gamma_2])$  (see Figure 18). Hence we have that

$$\begin{aligned} [Y_{2;3}, Y_{1;3}^{-1}] &= Y_{2;3}^{-1}Y_{1;3}Y_{2;3}Y_{1;3}^{-1} \\ &= s_2([\gamma_2^{-1}])s_2([\gamma_1^2\gamma_2]) \\ &= s_2([\gamma_1^2\gamma_2][\gamma_2^{-1}]) \\ &= s_2([\gamma_1^2]) \\ &= t_{c_1}t_{c_2}, \end{aligned}$$

where  $c_1$  and  $c_2$  are simple closed curves as shown in Figure 18. Similar to (1), we have that  $[Y_{2;3}, Y_{1;3}^{-1}]$  is a BSCC map of type (1, 2).

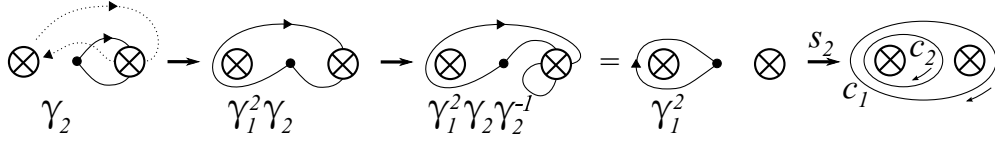


FIGURE 18.

(3) We have that

$$\begin{aligned} Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{1;2}Y_{1;3}Y_{2;3} &= Y_{2;3}^{-1}Y_{1;3}^{-1}s_1([\gamma_1])Y_{1;3}Y_{2;3} \\ &= Y_{2;3}^{-1}s_1([\gamma_2\gamma_1\gamma_2^{-1}])Y_{2;3} \\ &= s_1([\gamma_1^{-1}]) \\ &= Y_{1;2}^{-1} \end{aligned}$$

(see Figure 19 (a)). Hence we have that  $[Y_{1;2}, Y_{1;3}Y_{2;3}] = Y_{1;2}^{-1}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{1;2}Y_{1;3}Y_{2;3} = Y_{1;2}^{-2}$ . Similarly, we have that

$$\begin{aligned} Y_{1;2}^{-1}Y_{3;2}^{-1}Y_{1;3}Y_{3;2}Y_{1;2} &= Y_{1;2}^{-1}Y_{3;2}^{-1}s_1([\gamma_2])Y_{3;2}Y_{1;2} \\ &= Y_{1;2}^{-1}s_1([\gamma_1^{-1}\gamma_2^{-1}\gamma_1])Y_{1;2} \\ &= s_1([\gamma_2^{-1}]) \\ &= Y_{1;3}^{-1} \end{aligned}$$

(see Figure 19 (b)). Hence we have that  $[Y_{1;3}, Y_{3;2}Y_{1;2}] = Y_{1;3}^{-2}$ . Similar to (1), we have that  $Y_{1;2}^{-2}$  and  $Y_{1;3}^{-2}$  are BSCC maps of type (1, 2).

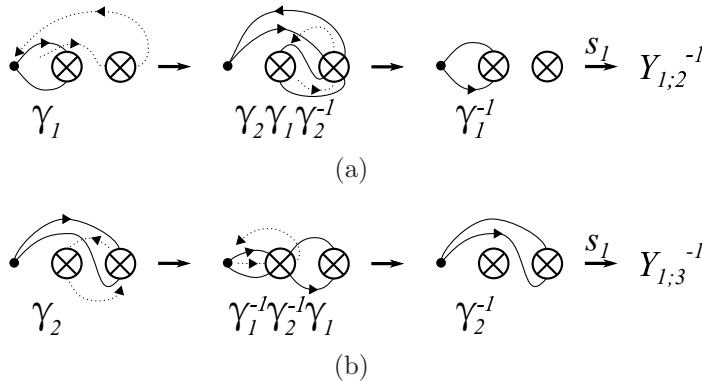


FIGURE 19.

- (4) In  $\mathcal{I}(N_g)$ , it is clear that  $[Y_{1;2}, Y_{3;4}]$  and  $[Y_{2;3}^{-2}Y_{1;3}Y_{2;3}^2, Y_{2;4}]$  are equal to 1. Note that  $Y_{2;3}^{-2}Y_{1;3}Y_{2;3}^2 = s_1([\gamma_1^{-2}\gamma_2\gamma_1^2])$  (see Figure 20).

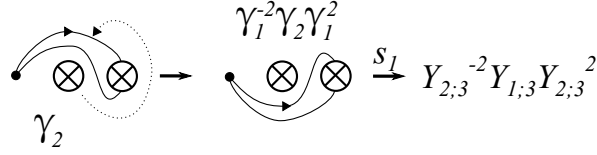
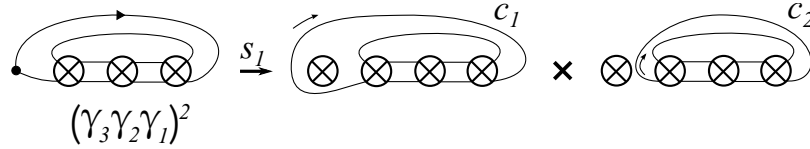


FIGURE 20.

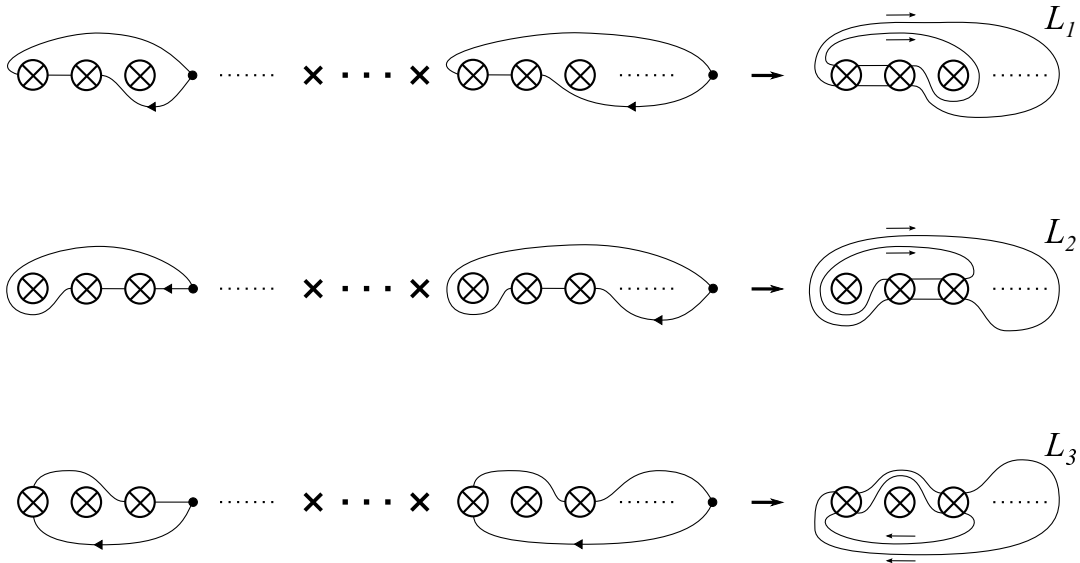
- (5) We have that  $(Y_{1;2}Y_{1;3}Y_{1;4})^2 = s_1([\gamma_3\gamma_2\gamma_1]^2) = t_{c_1}t_{c_2}$ , where  $c_1$  and  $c_2$  are simple closed curves as shown in Figure 21. Note that  $t_{c_1}$  is a BSCC map of type (1, 2) if  $g \geq 5$  and a BSCC map of type (2, 1) if  $g = 4$ . Since  $c_2$  bounds a Möbius band,  $t_{c_2}$  is trivial. Hence we have that  $(Y_{1;2}Y_{1;3}Y_{1;4})^2$  is a BSCC map of type (1, 2) if  $g \geq 5$  and a BSCC map of type (2, 1) if  $g = 4$ .

FIGURE 21. The black cross “ $\times$ ” means the composition of  $t_{c_1}$  with  $t_{c_2}$ .

- (6) By the proof of Lemma 3.1 in [17], we have  $Y_{j;i}^{-1}Y_{i;j} = Y_{j;i}Y_{i;j}^{-1} = T_{i,j}^2$ , for  $1 \leq i < j \leq g$ . Hence we have that  $(Y_{2;1}^{-1}Y_{1;2}Y_{3;2}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{3;1})^2 = (T_{1,2}^2T_{2,3}^2T_{1,3}^{-2})^2$ .

**Lemma 5.2.** *We have followings (see Figure 22).*

- (a)  $T_{1,2}^2 = L_1 Y_{3;1} Y_{3;2}$ , where  $L_1 = (Y_{g;1}Y_{g;2}Y_{g;3}^2 \cdots Y_{g;g-1}^2) \cdots (Y_{4;1}Y_{4;2}Y_{4;3}^2)$ .
- (b)  $T_{2,3}^2 = L_2 Y_{1;2} Y_{1;3}$ , where  $L_2 = (Y_{g;1}^2 Y_{g;2} Y_{g;3} Y_{g;4}^2 \cdots Y_{g;g-1}^2) \cdots (Y_{4;1}^2 Y_{4;2} Y_{4;3})$ .
- (c)  $T_{1,3}^{-2} = L_3 Y_{2;3} Y_{2;1}$ , where  $L_3 = (Y_{g;3}^{-1} Y_{g;1} Y_{g;2}^2 \cdots Y_{g;g-1}^2) \cdots (Y_{4;3}^{-1} Y_{4;1} Y_{4;2}^2 Y_{4;3}^2)$ .

FIGURE 22. In this figure, and also Figures 24, 25 and 27, the black crosses “ $\times$ ” mean compositions of the crosscap pushing maps.

We have

$$\begin{aligned}
& Y_{3;2}^{-2} Y_{3;1}^{-1} (Y_{2;1}^{-1} Y_{1;2} Y_{3;2} Y_{2;3}^{-1} Y_{1;3}^{-1} Y_{3;1})^2 Y_{3;1} Y_{3;2}^2 \\
&= Y_{3;2}^{-2} Y_{3;1}^{-1} \cdot L_1 Y_{3;1} Y_{3;2} \cdot Y_{3;2} Y_{2;3}^{-1} \cdot L_3 Y_{2;3} Y_{2;1} \\
&\quad \cdot Y_{2;1}^{-1} Y_{1;2} \cdot L_2 Y_{1;2} Y_{1;3} \cdot Y_{1;3}^{-1} Y_{3;1} \cdot Y_{3;1} Y_{3;2}^2 \\
&= R_1 R_3 R_2 Y_{1;2}^2 Y_{3;1}^2 Y_{3;2}^2,
\end{aligned}$$

where  $R_1 = Y_{3;2}^{-2} Y_{3;1}^{-1} L_1 Y_{3;1} Y_{3;2}^2$ ,  $R_2 = Y_{1;2} L_2 Y_{1;2}^{-1}$  and  $R_3 = Y_{2;3}^{-1} L_3 Y_{2;3}$  (see Figure 23). Similar to (1), since  $Y_{1;2}^2$ ,  $Y_{3;1}^2$  and  $Y_{3;2}^2$  are BSCC maps of type (1, 2), it is suffice to show that  $R_1 R_3 R_2$  is a product of BSCC maps. Let  $d_0$ ,  $d_1$ ,  $d_2$ ,  $d_3$  and

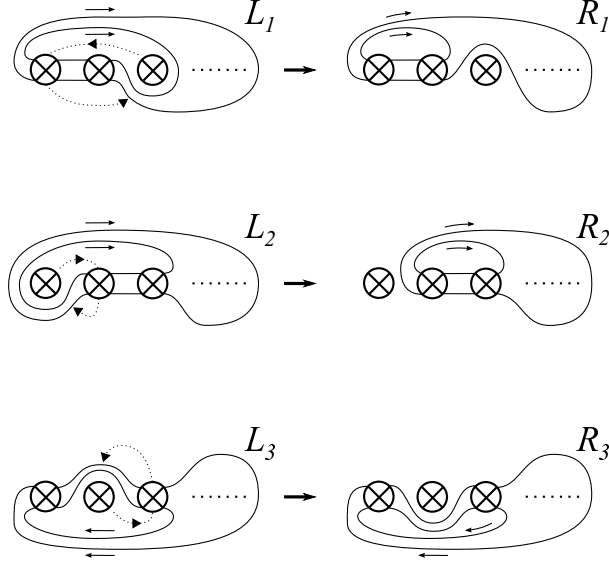


FIGURE 23.

$d_4$  be simple closed curves as shown in Figure 24. Note that  $t_{d_0}$  is a BSCC map of type (1, 3) or (1,  $g-3$ ), and  $t_{d_3}$  is a BSCC map of type (2, 1). In addition, since  $d_4$  bounds a Möbius band, we have  $t_{d_4} = 1$ . Let  $R_{32} = t_{d_1} t_{d_2}$ . By the lantern relation, we have that  $R_{32}$  is a product of  $R_3 R_2$  and  $t_{d_0}$ . Let  $R_{132} = t_{d_3} t_{d_4}$ . By the lantern relation, we have that  $R_{132}$  is a product of  $R_1 R_{32}$  and  $t_{d_0}$ . Hence we have that  $R_{132}$  is a product of  $R_1 R_3 R_2$  and  $t_{d_0}$ . Therefore  $R_1 R_3 R_2$  is a product of BSCC maps of type one and a BSCC map of type (2, 1). In particular, if  $g = 4$ , since  $t_{d_0}$  is trivial, we have that  $R_1 R_3 R_2$  is a BSCC map of type (2, 1), and if  $g \geq 5$ , by Lemma 2.2, we have that  $R_1 R_3 R_2$  is a product of BSCC maps of type (1, 2).

(7) We have

$$\begin{aligned}
T_{1,2,3,4}^{-1} T_{1,2,3,4}'^{-1} &= \prod_{5 \leq m \leq g} s_m([\gamma_1^{-1} \gamma_2^{-1} \gamma_3^{-1} \gamma_4^{-1} \gamma_5^{-2} \gamma_6^{-2} \cdots \gamma_{m-1}^{-2}]) \\
&= \prod_{5 \leq m \leq g} Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2} Y_{m;5}^{-2} Y_{m;4}^{-1} Y_{m;3}^{-1} Y_{m;2}^{-1} Y_{m;1}^{-1}.
\end{aligned}$$

(see Figure 25). Hence we have

$$T_{1,2,3,4}^2 \left( \prod_{5 \leq m \leq g} Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2} Y_{m;5}^{-2} Y_{m;4}^{-1} Y_{m;3}^{-1} Y_{m;2}^{-1} Y_{m;1}^{-1} \right) = T_{1,2,3,4} T_{1,2,3,4}'^{-1}.$$

Thus it is a BP map of type (1, 1).

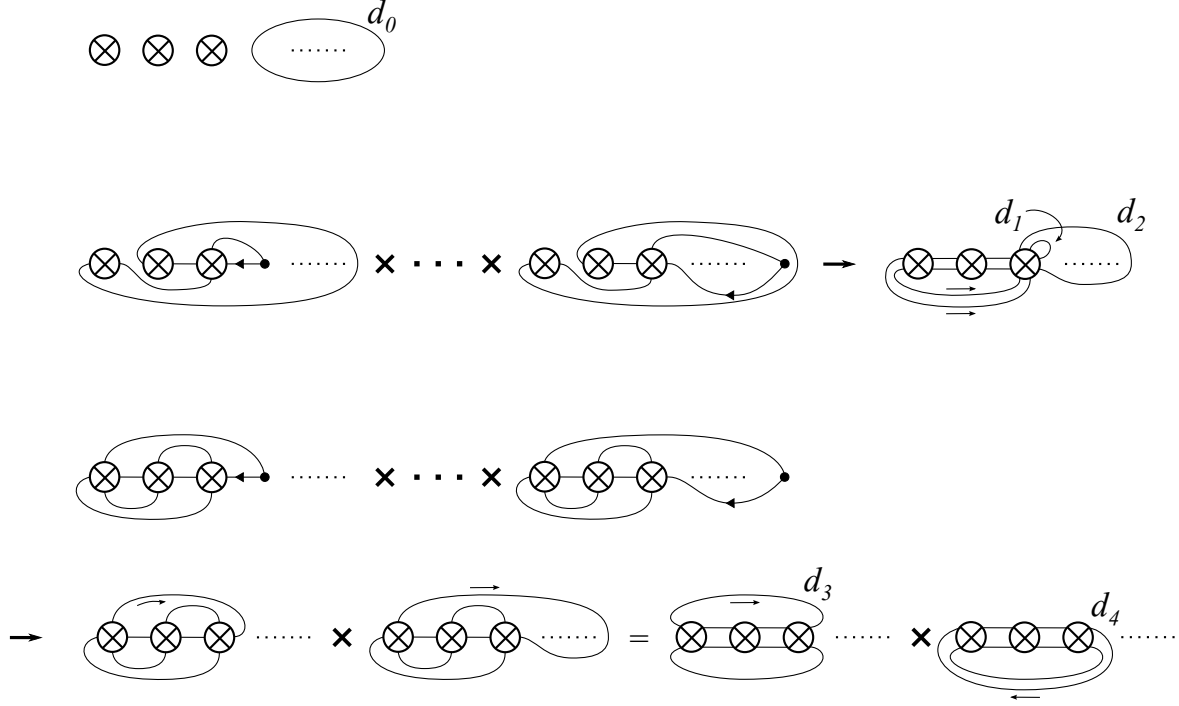


FIGURE 24.

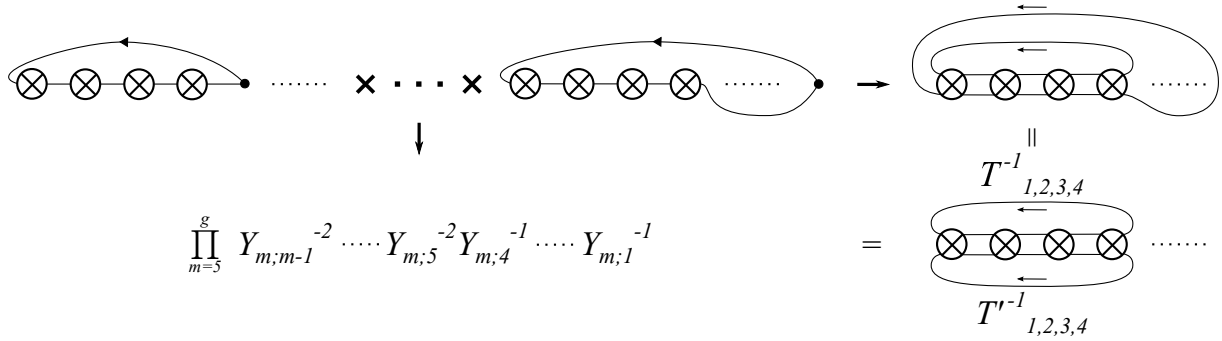


FIGURE 25.

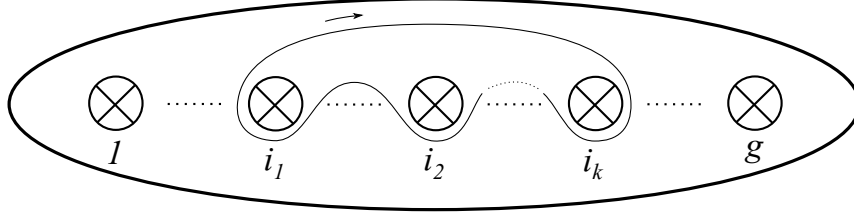
Thus we complete the proof.  $\square$

## APPENDIX A.

In this appendix, we show the equation (I) in Section 2 and the equation (II) in Section 4. For (I), it suffices to show the following lemma.

**Lemma A.1.** *For  $1 \leq i_1 < i_2 < \cdots < i_k \leq g$ , let  $c_{i_1, \dots, i_k}$  be a simple closed curve on  $N_g$  as shown in Figure 26. Then for  $2 \leq h \leq g$  we have*

$$t_{c_{1, \dots, h}} = (t_{c_{1,2}} \cdots t_{c_{1,h-1}} t_{c_{1,h}}) \cdots (t_{c_{h-2,h-1}} t_{c_{h-2,h}}) (t_{c_{h-1,h}}).$$

FIGURE 26. The curve  $c_{i_1, \dots, i_k}$  on  $N_g$ .

*Proof.* We first note that  $t_{c_{i,j}} = Y_{i,j}^2$ . We see

$$\begin{aligned}
 t_{c_{1, \dots, h}} &= (t_{c_{1, \dots, h}} t_{c_{2, \dots, h}}^{-1}) \cdots (t_{c_{h-2, h-1, h}} t_{c_{h-1, h}}^{-1}) (t_{c_{h-1, h}} t_{c_h}^{-1}) \\
 &= (s_1([\gamma_{h-1}^2 \cdots \gamma_1^2])) \cdots (s_{h-2}([\gamma_{h-1}^2 \gamma_{h-2}^2])) (s_{h-1}([\gamma_{h-1}^2])) \\
 &= (Y_{1;2}^2 \cdots Y_{1;h-1}^2 Y_{1;h}^2) \cdots (Y_{h-2;h-1}^2 Y_{h-2;h}^2) (Y_{h-1;h}^2) \\
 &= (t_{c_{1,2}} \cdots t_{c_{1,h-1}} t_{c_{1,h}}) \cdots (t_{c_{h-2,h-1}} t_{c_{h-2,h}}) (t_{c_{h-1,h}}).
 \end{aligned}$$

Thus we obtain the claim.  $\square$

**Example A.2.** For  $1 \leq i \leq g-1$  we have

$$\begin{aligned}
 Y_{g;i} &= (Y_{1;2}^2 \cdots Y_{1;g-1}^2 Y_{1;i}^{-1} Y_{1;g}) \cdots (Y_{i-1;i}^2 \cdots Y_{i-1;g-1}^2 Y_{i-1;i}^{-1} Y_{i-1;g}) \\
 &\quad \cdot (Y_{i+1;i+2}^2 \cdots Y_{i+1;g-1}^2 Y_{i+1;i}^{-1} Y_{i+1;g}) \cdots (Y_{g-2;g-1}^2 Y_{g-2;i}^{-1} Y_{g-2;g} Y_{g-2;i}^2) \\
 &\quad \cdot (Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2) Y_{i;g}.
 \end{aligned}$$

*Proof.* Note that  $Y_{g;i} Y_{i;g}^{-1} = T_{i,g}^2$ . We see

$$\begin{aligned}
 T_{i,g}^2 &= \prod_{1 \leq m \leq i-1} s_m([\gamma_{g-1} \gamma_{i-1}^{-1} \gamma_{g-2}^2 \cdots \gamma_m^2]) \prod_{i+1 \leq m \leq g-2} s_m([\gamma_i^2 \gamma_{g-1} \gamma_i^{-1} \gamma_{g-2}^2 \cdots \gamma_m^2]) \\
 &\quad \cdot s_{g-1}([\gamma_i^2 \gamma_{g-1} \gamma_i^{-1}]) \\
 &= \prod_{1 \leq m \leq i-1} (Y_{m;m+1}^2 \cdots Y_{m;g-1}^2 Y_{m;i}^{-1} Y_{m;g}) \prod_{i+1 \leq m \leq g-2} (Y_{m;m+1}^2 \cdots Y_{m;g-1}^2 Y_{m;i}^{-1} Y_{m;g} Y_{m;i}^2) \\
 &\quad \cdot (Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2)
 \end{aligned}$$

(see Figure 27). Thus we obtain the claim.  $\square$

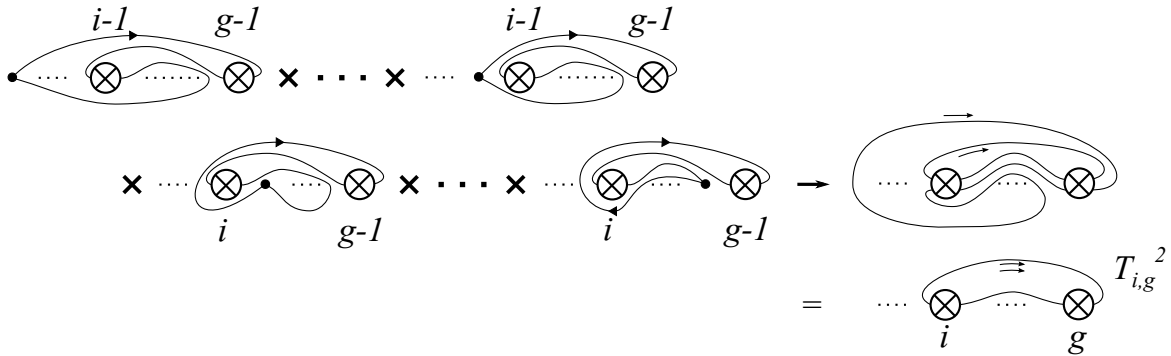


FIGURE 27.

Thus we obtain the equation (II).



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